

Quantum Probabilistic Structures in Competing Lizard Communities*

Diederik Aerts¹, Marek Czachor², Maciej Kuna³, Barry Sinervo⁴ and Sandro Sozzo¹

¹ *Center Leo Apostel for Interdisciplinary Studies
Department of Mathematics and Department of Psychology
Brussels Free University, Brussels, Belgium
Emails: diraerts@vub.ac.be, ssozzo@vub.ac.be*

² *Department of Theoretical Physics and Quantum Information
Politechnika Gdańska, Gdańsk, Poland
Email: mczachor@pg.gda.pl*

³ *Department of Probability and Biomathematics
Politechnika Gdańska, Gdańsk, Poland
Email: maciek@mif.pg.gda.pl*

⁴ *Department of Biology and Evolutionary Biology
University of California, Santa Cruz, California, USA
Email: lizardrps@gmail.com*

Abstract

Almost two decades of research on the use of the mathematical formalism of quantum theory as a modeling tool for entities and their dynamics in domains different from the micro-world has now firmly shown the systematic appearance of quantum structures in aspects of human behavior and thought, such as in cognitive processes of decision-making, and in the way concepts are combined into sentences. In this paper, we extend this insight to animal behavior showing that a quantum probabilistic structure models the mating competition of three side-blotched lizard morphs. We analyze a set of experimental data collected from 1990 to 2011 on these morphs, whose territorial behavior follows a cyclic rock-paper-scissors (RPS) dynamics. Consequently we prove that a single classical Kolmogorovian space does not exist for the lizard's dynamics, and elaborate an explicit quantum description in Hilbert space faithfully modeling the gathered data. This result is relevant for population dynamics as a whole, since many systems, e.g. the so called plankton paradox situation, are believed to contain elements of cyclic competition.

*This paper is dedicated to the memory of our dear colleague and friend Prof. Bart D'Hooghe, who recently passed away, after collaborating with us for several years and contributing to the first draft of this manuscript.

Keywords: lizard communities, RPS game, non-Kolmogorovian probabilities, quantum structures

1 Introduction

The use of the mathematical formalism of quantum mechanics as a modeling instrument in domains different from the micro-world has led to remarkable results in recent years and is now an active and emergent research field in itself. In cognitive science, e.g. concept theory and decision theory, in economics, e.g. finance and behavioral economics, and in computer science, e.g. semantic theories, information retrieval and artificial intelligence, several situations have been identified where the application of classical structures is problematical, while quantum structures successfully model these situations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The overall approach of using quantum structures in domains different from the micro-world has indicated that it is not the existence of microscopic quantum processes that makes a quantum-theoretical modeling so efficient but, rather, the fact that the systems considered in these disciplines unexpectedly share some typical quantum features, such as indeterminism of the non-classical type, contextuality, interference, entanglement and superposition. It has been found that these quantum aspects are quite systematically present in human cognition, such as in processes of decision-making [1, 5, 6, 11, 12, 14] and in the dynamics of how humans use and combine concepts [4, 8, 13, 15]. These findings have now been published in top peer-reviewed journals, such as ‘Journal of Physics A’ [3], ‘Proceedings of the Royal Society B’ [6], ‘Journal of Mathematical Psychology’ [8], ‘Psychological Review’ [11], ‘Topics in Cognitive Science’ [13], and this year in a target article and commentary in ‘Brain and Behavioral Sciences’ [14, 15], which shows that the results have been generally peer acknowledged in a broad field of cognitive science. Also the findings related to economics and computer science are all connected to human thought and behavior.

This article looks into another challenging question, namely the question of whether quantum structures can be found in animal behavior and their dynamics as well. We will discuss an example that provides an affirmative answer to this question. More specifically, we will reveal the presence of quantum structures in the mating competition of three male morphs of a specific lizard species. In this way, we are extending this quantum structure area of research into human thought and behavior and its results to include animal behavior and the evolutionary analysis of these competitive games.

The first step leading to the result we put forward in the present article was related to our study of biological evolution based on a specific situation involving the rock-paper-scissors (RPS) game as an example. We observed that when the RPS game was regarded as a coincidence experiment, it allowed for violations of Bell’s inequalities [16, 17]. This was an intriguing matter indeed, because it has been shown in the foundations of quantum mechanics that a potential violation of Bell’s inequalities is indicative of the presence of a non-Kolmogorovian probability model, i.e. a probability model that does not satisfy Kolmogorov’s axioms for classical probability, and hence possibly a quantum probability model [18, 19, 20, 21]. However, the wake-up call with respect to the importance of this insight came only when we noticed that the RPS game had been used as a modeling scheme for specific types of dynamical situations in population ecology, including the

famous ‘paradox of the plankton’, where it is referred to as ‘cyclic competition’ or, a generalization of it, ‘multiple competition’ [22, 23, 24, 25, 26]. Also situations of competing lizard species were studied intensively by considering cyclic competition as a fundamental aspect of their dynamics. More specifically, one of us discovered an RPS strategy in the mating behavior of the side-blotched lizard species *Uta stansburiana* [27, 28]. Males have either orange, blue or yellow throats and each type follows a fixed, heritable mating strategy. As in the RPS game, where scissors cut paper, rock crushes scissors, and paper wraps rock, the three-morph mating system in the side-blotched lizard is such that the wide-ranging ultradominant strategy of orange males is defeated by the sneaker strategy of yellow males, which is in turn defeated by the mate-guarding strategy of blue males; the orange strategy defeats the blue strategy, to complete the dynamic cycle. This *lizard game* presents a stable pattern in the replicator dynamics where the dynamical system follows closed orbits around the mixed strategy Nash equilibrium [27, 28, 29, 30, 31].

It is this lizard ecosystem that is the focus of our study, and for which we construct an explicit quantum theoretic representation, making use of the rules of the quantum formalism to calculate probabilities, and which faithfully models the experimental data gathered by one of us over the last two decades. Rather than directing our attention to the abstract rock paper scissors dynamics, and revealing the non-Kolmogorovian probability structure for this theoretical situation, we want to focus directly on the experimental data themselves, gathered from the real-life lizard situation dynamics. In Section 2, we prove that these real-life data cannot be modeled within a probability structure satisfying the axioms of Kolmogorov. In Section 3, we work out an explicit quantum mechanical representation in complex Hilbert space for the probabilities in the pay-off matrices of the lizard real-life situation, which allows us to recover in a faithful way the experimentally collected probabilities. In Section 4 we elaborate and give hints for future investigation with respect to our result.

To end, we wish to remark that the evidence provided in this paper of quantum probability models naturally appearing in the lizard dynamics opens new and interesting scenarios for the employment of quantum structures to describe animal behavior and interactions, and constitutes a strong support toward the application of these structures to ecological and biological systems and their evolutionary dynamics.

2 The non-classical probabilistic nature of the lizard dynamics

Before we proceed to provide formal proof of the non-Kolmogorovian nature of the probabilities involved in the lizard situation, let us use the experience and insights gained over decades in the foundations of quantum physics to identify the origin of this non-classical probabilistic structure. What we said in Section 1 for the cases of quantum structures in cognition also applies to the lizard dynamics, in that the origin of the non-classicality should not be sought in the existence of some microscopic processes of a quantum-mechanical type. This does not mean that its origin is not fundamental; on the contrary, since it lies in the fact that evolutionary games and quantum phenomena share the property of being context dependent (see Figure 1). It has been known in the quantum mechanics community for at least 30 years that contextuality is one of the fundamental sources of the quantumness of quantum probability [19, 20, 21, 32, 33]. One can even say that

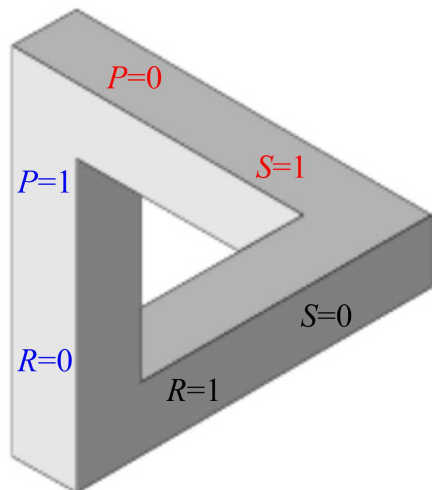


Figure 1: Contextuality of the RPS game

the role contextuality plays in probability is similar to that of curvature in the geometry of non-Euclidean spaces. Kolmogorovity in probability is like Euclideanity in geometry. But as non-Euclideanity is not an indication of gravity, non-Kolmogorovity is not an indication of quanta, even though geometry becomes non-Euclidean in the presence of gravity, and quantum statistics is non-Kolmogorovian.

The question arises how the quantum aspect of a classical game could have been overlooked so far, and why it is important. The catch is that standard game theory concentrates on probabilities of strategies, treating probabilities of “packs of cards”, inherent in pay-off matrices, as given and not as entangling with the players’ behaviors. But it is the latter probabilities that may reveal non-Kolmogorovian properties in classical games. In the RPS game, each matrix element of a pay-off matrix is evaluated in a different probability space. As long as there are no processes that entangle these two levels of probability, one can ignore implicit non-Kolmogorovity of pay-offs. However, if the two levels do get entangled, the non-Kolmogorovity will become essential.

In classical evolutionary game theory, the elements of the pay-off matrix are assumed to be time invariant and evolution of the system takes place as frequency-dependent fitness changes, thereby changing the relative success and the probability of encountering each strategy over time. In evolutionary biology, the strategies can be identified with morphs, and many species exhibit color polymorphisms associated with alternative male reproductive strategies [28, 31, 34]. The prevalence of multiple morphs is a challenge to evolutionary theory because a single strategy should prevail unless morphs have exactly the same fitness or a fitness advantage when rare. One of us has shown in several papers that the three color morphs of side-blotched lizards, *Uta stansburiana*, follow RPS-like dynamics [28, 29, 30, 34, 35]. More precisely, males have either orange (*O*), blue (*B*) or yellow (*Y*) throats and each type follows a fixed mating strategy, as follows:

- (i) Orange-throated males are strongest and do not form strong pair bonds; instead, they fight

blue-throated males for their females. Yellow-throated males, however, manage to copulate with females in the orange male harems. The large size and aggression is caused by high testosterone production [36].

(ii) Blue-throated males are smaller in size and form strong pair bonds. While they are outcompeted by orange-throated males, they can defend against yellow-throated ones via co-operation with other blue-throated neighbors. Because blue-throated males produce less testosterone, they are not as strong as the orange-throated males, but it gives them the advantage of being less aggressive and able to form strong pair bonds, and also engage in territorial co-operation with neighboring blue-throated males [30].

(iii) Yellow-throated males are smallest, and their coloration mimics females. This enables them to approach females in the harems of orange-throated males and mate when the latter are distracted. This is less likely to work with a female that has bonded with a blue-throated male, and by virtue of his vigilant co-operative blue-throated male partner.

Points (i)-(iii) can be summarized as “ O beats B , B beats Y , and Y beats O ”, which is similar to the RPS rules, hence O provides a context for B which, in turn, provides a context for Y , which provides a context for O . Thus, the biology of the RPS lizards hints at a deeper underlying contextuality for the interactions beyond the simple pairwise dynamics that can be encapsulated in pairwise elements of the RPS pay-off matrix.

In order to perform the test for Kolmogorovity, we have to rephrase the lizard game as a two-system correlation experiment, and we will do so with the help of two imaginary participants, Alice and Bob. So, let us now consider the three throat-color morphs of side-blotched lizards, where rare O , B , Y define R , P , S of Alice, and common O , Y , B define R , P , S of Bob. The rules of the game are as follows: If Alice chooses R and Bob chooses P , then $(R, P) = (-, +)$. Analogously, $(R, S) = (+, -)$, $(P, S) = (-, +)$, $(P, R) = (+, -)$, $(S, R) = (-, +)$, $(S, P) = (+, -)$.

We employ the frequencies from 1990-2011 collected using the same methods and from the same population as in the original study on lizards [28] (Table 1). Looking at Figure 2, we find that there are no games where Alice and Bob play the same strategy (two morphs cannot be simultaneously rare or simultaneously common). The conditional probabilities $p(i_A = +|j_B)$ that Alice playing an i th strategy wins if Bob plays a j th strategy, are derived from a published experiment [35] (data from 2001–2003) using the multiple linear regression methods for pay-off matrix estimation [34],

$$\begin{aligned}
& \begin{pmatrix} & 0.161616 & 0.574257 \\ 0.571429 & & 0.168317 \\ 0.204082 & 0.575758 & \end{pmatrix} \\
&= \begin{pmatrix} & p(O_A = +|Y_B) & p(O_A = +|B_B) \\ p(Y_A = +|O_B) & & p(Y_A = +|B_B) \\ p(B_A = +|O_B) & p(B_A = +|Y_B) & \end{pmatrix} \\
&= \begin{pmatrix} & p(Y_B = -|O_A) & p(B_B = -|O_A) \\ p(O_B = -|Y_A) & & p(B_B = -|Y_A) \\ p(O_B = -|B_A) & p(Y_B = -|B_A) & \end{pmatrix}.
\end{aligned}$$

Note that the conditioning is not by the *result* obtained by Bob, but by the *strategy* he plays, i.e. by

Year	O frequency	B frequency	Y frequency	Alice's strategy	Bob's strategy
1990	0.118644068	0.610169492	0.271186441	O	B
1991	0.177419355	0.728813559	0.093767086	Y	B
1992	0.291666667	0.401869159	0.306464174	O	B
1993	0.247619048	0.42	0.332380952	O	B
1994	0.158878505	0.415841584	0.425279911	O	Y
1995	0.160377358	0.484848485	0.354774157	O	B
1996	0.150684932	0.666666667	0.182648402	O	B
1997	0.266666667	0.25	0.483333333	B	Y
1998	0.132075472	0.5	0.367924528	O	B
1999	0.088235294	0.666666667	0.245098039	O	B
2000	0.094339623	0.627906977	0.277753401	O	B
2001	0.154761905	0.432098765	0.41313933	O	B
2002	0.204545455	0.175	0.620454545	B	Y
2003	0.160377358	0.424242424	0.415380217	O	B
2004	0.027777778	0.655737705	0.316484517	O	B
2005	0.017241379	0.58490566	0.39785296	O	B
2006	0.134615385	0.270833333	0.594551282	O	Y
2007	0.351351351	0.088235294	0.560413355	B	Y
2008	0.6	0.157894737	0.242105263	B	O
2009	0.385714286	0.301587302	0.312698413	B	O
2010	0.633333333	0.266666667	0.1	Y	O
2011	0.808510638	0.170212766	0.021276596	Y	O

Table 1: Experimental data on the relative abundance of the three male morphs of the side-blotched lizard *Uta stansburiana* collected from 1990 to 2011.

the *context* in which Alice is supposed to win. The diagonal elements correspond to cases where the players play the same strategy, which does not occur in our Alice–Bob reformulation, so we will skip them. Now, let us assume that the above probabilities can be modeled on a single probability space. So, there exists a measurable space Λ and a probability measure μ on Λ . The events, say, $O_A = +$ and $O_A = -$, must correspond to sets $\mathcal{O}_A \subset \Lambda$ and $\mathcal{O}'_A = \Lambda \setminus \mathcal{O}_A$, respectively. Similar rules hold for the remaining random variables. Then, by our assumption, the joint probability that rare O fails whereas common Y succeeds in siring young is $p(O_A = - \cap Y_B = +) = \mu(\mathcal{O}'_A \cap \mathcal{Y}_B) = 0.702888$. The probability that the two morphs both succeed is $p(O_A = + \cap Y_B = +) = \mu(\mathcal{O}_A \cap \mathcal{Y}_B) = 0.135496$, hence

$$\mu(\mathcal{Y}_B) = \mu(\mathcal{O}'_A \cap \mathcal{Y}_B) + \mu(\mathcal{O}_A \cap \mathcal{Y}_B) = 0.838384 = p(Y_B = + | O_A). \quad (1)$$

However, an analogous reasoning performed for B and Y implies $p(B_A = + \cap Y_B = +) = \mu(\mathcal{B}_A \cap \mathcal{Y}_B) = 0.244261$, $p(B_A = - \cap Y_B = +) = \mu(\mathcal{B}'_A \cap \mathcal{Y}_B) = 0.179981$, and $\mu(\mathcal{Y}_B) = 0.424242 = p(Y_B = + | B_A)$. This proves the nonexistence of μ and Λ . The apparently obvious formulas of the form

$$p(Y_B = + | B_A) = p(B_A = + \cap Y_B = +) + p(B_A = - \cap Y_B = +) \quad (2)$$

link probabilities at different levels of contextuality. The right side refers to results of measure-

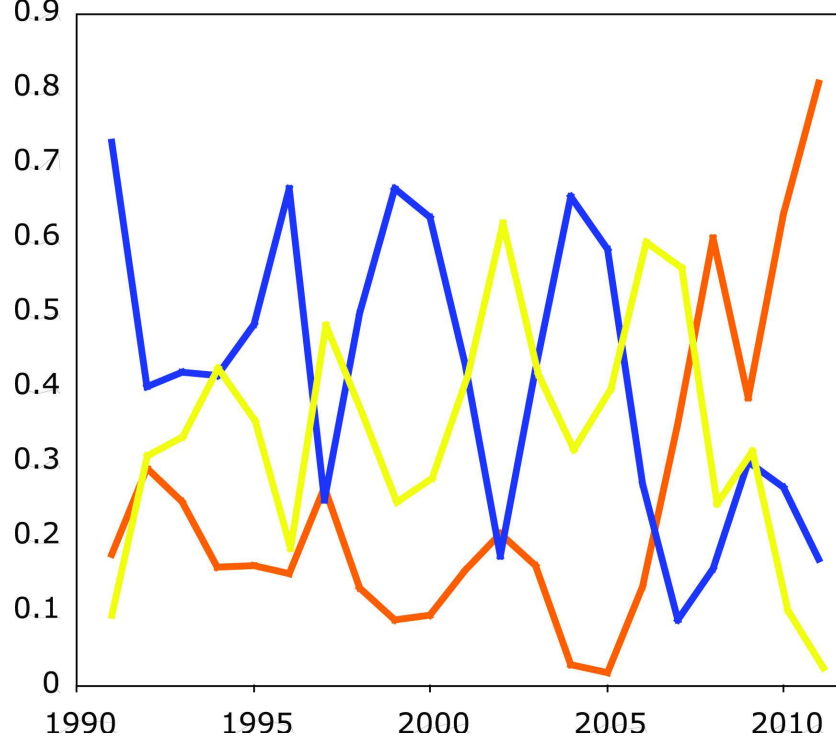


Figure 2: Observed male color morph frequencies for the lizard *Uta stansburiana* at Los Banos, California, U.S.A. Highest and lowest populations define, common and rare morphs, respectively.

ments of random variables, but the left one refers only to the context. Actually, (2) is a formal way of *defining* context by means of types of measurements performed by the other player.

As we can see, the contradiction is very elementary, $p(Y_B = +|B_A) \neq p(Y_B = +|O_A)$, and certainly typical of all nontrivial evolutionary games. Since this contradiction follows from the hypothesis that one Kolmogorovian probability space exists for the probabilities appearing in the lizard dynamics, we have disproved this hypothesis. Hence, at least different mutually incompatible probability spaces are needed. In the next section we construct an explicit quantum probabilistic model for the experimental data of the lizard dynamics considered.

3 A quantum representation of the lizard dynamics

In this section we prove the less obvious property, namely that the situation is not only non-Kolmogorovian, but also quantum. To do so, it is enough to find a density operator ρ and, for any pair of morphs i_A and j_B , the set of four positive operator valued measures (POVM) [37] $E_{i_A \cap j_B}$ satisfying $p_{i_A \cap j_B} = \text{Tr}(\rho E_{i_A \cap j_B})$ and

$$p_{i_A \cap j_B} + p_{i'_A \cap j_B} + p_{i_A \cap j'_B} + p_{i'_A \cap j'_B} = 1.$$

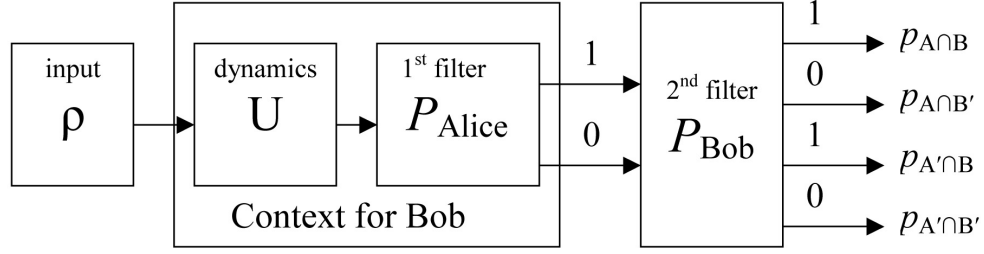


Figure 3: The physical structure of POVM. Since $\text{Tr}(\rho U P_{\text{Alice}} P_{\text{Bob}} P_{\text{Alice}} U^\dagger) = \text{Tr}(\rho \tilde{P}_{\text{Alice}} U P_{\text{Bob}} U^\dagger \tilde{P}_{\text{Alice}})$, where $\tilde{P}_{\text{Alice}} = U P_{\text{Alice}} U^\dagger$, an identical effect is obtained by replacing the first filter P_{Alice} by \tilde{P}_{Alice} , and including the dynamics between the filters. Note that U refers to an internal dynamics of the measuring device and not to the dynamics of probabilities themselves. The latter would evolve by means of $\rho(t)$.

The form of POVM that is sufficiently general for our purposes physically corresponds to experimental devices consisting of two filters [38], supplemented by unitary dynamics before or after the first filter, see Figure 3. The dynamics and the first filter define the context for the second filter. Denote $a = p_{i_A \cap j_B}$, $b = p_{i_A \cap j'_B}$, $c = p_{i'_A \cap j_B}$, $d = p_{i'_A \cap j'_B}$, and $\langle 1| = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$. Let $|2\rangle$, $|3\rangle$ and $|4\rangle$ be any vectors that together with $|1\rangle$ form an orthonormal basis. The density operator $\rho = |1\rangle\langle 1|$, and the commuting projectors

$$P_{\text{Alice}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_{\text{Bob}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$P'_{\text{Alice}} = \mathbb{I} - P_{\text{Alice}}$, $P'_{\text{Bob}} = \mathbb{I} - P_{\text{Bob}}$, imply the resolution of identity

$$P_{\text{Alice}} P_{\text{Bob}} + P_{\text{Alice}} P'_{\text{Bob}} + P'_{\text{Alice}} P_{\text{Bob}} + P'_{\text{Alice}} P'_{\text{Bob}} = \mathbb{I},$$

and the correct probabilities

$$\begin{aligned} a &= \text{Tr}(\rho P_{\text{Alice}} P_{\text{Bob}}), \\ b &= \text{Tr}(\rho P_{\text{Alice}} P'_{\text{Bob}}), \\ c &= \text{Tr}(\rho P'_{\text{Alice}} P_{\text{Bob}}), \\ d &= \text{Tr}(\rho P'_{\text{Alice}} P'_{\text{Bob}}). \end{aligned}$$

Now let a' , b' , c' , d' be *any* other four probabilities, and $\langle 1'| = (\sqrt{a'}, \sqrt{b'}, \sqrt{c'}, \sqrt{d'})$. We again complete $|1'\rangle$ to an orthonormal basis by means of $|2'\rangle$, $|3'\rangle$, $|4'\rangle$. Then

$$\begin{aligned} a' &= \text{Tr}(|1'\rangle\langle 1'| P_{\text{Alice}} P_{\text{Bob}}) \\ &= \text{Tr} \left(\underbrace{\sum_{i=1}^4 |i'\rangle\langle i'|}_{U^\dagger} \underbrace{|1\rangle\langle 1|}_\rho \underbrace{\sum_{j=1}^4 |j\rangle\langle j|}_{U} P_{\text{Alice}} P_{\text{Bob}} \right) \end{aligned}$$

$$= \text{Tr}(\rho U P_{\text{Alice}} P_{\text{Bob}} U^\dagger).$$

U so derived automatically guarantees that

$$\begin{aligned} b' &= \text{Tr}(\rho U P_{\text{Alice}} P'_{\text{Bob}} U^\dagger), \\ c' &= \text{Tr}(\rho U P'_{\text{Alice}} P_{\text{Bob}} U^\dagger), \\ d' &= \text{Tr}(\rho U P'_{\text{Alice}} P'_{\text{Bob}} U^\dagger). \end{aligned}$$

Each of these POVMs is a projector. In this way one can encode an arbitrary number of quadruples of probabilities in a single ρ , similarly to what one does in quantum cryptography with infinitely many qubits encoded in a single two-dimensional state. The essence of contextuality lies in non-commutativity of the triple $(U, P_{\text{Alice}}, P_{\text{Bob}})$. This is why in a classical framework, where everything commutes, the trick with U will not work. Here, it works for all quadruples of probabilities. The quantum model of probability thus turns out to be general enough to include *all* RPS-type population games. It should be stressed here that a quantum model is not just a collection of classical ‘local’ theories [39]: The reasons for that are identical to those that make it impossible to interpret quantum mechanics as just a collection of various realizations of classical mechanics, in spite of the fact that any single quantum experiment is in fact based on the naive ‘frequency interpretation’ of probability. Quantum probability is an ingenious way of combining incompatible local probabilistic structures into a single global one. The link of quantum probability to manifolds is thus not accidental [40].

The true meaning of probabilistic quantumness becomes clear if one understands that in quantum probability all the contextual aspects refer to a *single* dynamical statistical system, be it microscopic or cognitive in nature. This is why density matrix equations are expected to have applications reaching far beyond quantum theory. In particular, non-linear soliton density matrix equations [41, 42, 43], due to their universality typical of soliton systems, were anticipated to have biological applications [44]. It is interesting that the authors of [44] were not aware at that time that standard replicator equations of game theory can be indeed cast into a quantum-probability von Neumann form [45, 46] where ρ is precisely of the type we have just employed in reconstructing the lizard probabilities. Still, for replicator equations, the corresponding $U = \mathbb{I}$, so all the probabilities of strategies may be regarded as belonging to a single context. A single probability space is thus enough for replicator dynamics, a fact explaining its Kolmogorovity. Generalization beyond a single probability space, performed at the level of strategies, would lead to non-standard population games. The so-called quantum-like games [10, 33], if extended to population dynamics, would belong to this category.

What we do in the present paper is more fundamental. The game we consider is completely classical. The quantum-probabilistic structure is present but dormant. Can it be activated? In principle it can, in cases where the dynamics couples “pack-of-cards” probabilities, implicit in pay-off matrices, with the probabilities of players’ strategies. This is what happens in games involving players with “an ace up their sleeve”, whose replicator-type equation can be derived directly from the Gafiychuk–Prykarpatsky von Neumann form [45, 46] generalized to non-product density matrices in the same way as in non-linear generalizations of quantum mechanics [47]. Surprisingly, games of this type are played by the lizards. In the absence of orange, and in the

presence of blue, yellow pulls an ace out of its sleeve and transforms into blue [29, 36]. Further research will require contextualities of a higher order. For example, blue is only highly successful against yellow in the context of low orange neighborhoods and other blue males [30]. Plankton communities will be even more complicated. Although these behaviors themselves do not violate classical evolutionary game theory, it is impossible to model the probability space using existing game theoretical models, so that quantum probability models involving a single density matrix are required.

4 The ideal RPS game and the ‘ace-up-one’s-sleeve’ replicator equation

We analyze in this section a proof of non-Kolmogorovity of the ideal RPS game, which sheds light on the logic of the general proof in Sections 2. We also elaborate our investigation exposed in Section 3 to construct the ‘ace-up-one’s-sleeve’ replicator equation.

Hence, we consider the ideal game, characterized by the following simple table of conditional probabilities,

$$\begin{array}{ccccc}
 & R & P & S & \text{Bob} \\
 \begin{array}{c} R \\ P \\ S \\ \text{Alice} \end{array} & & 0 & 1 & \\
 & 1 & & 0 & \\
 & 0 & 1 & &
 \end{array}
 = \left(\begin{array}{cc} p(R_A = +|P_B) & p(R_A = +|S_B) \\ p(P_A = +|R_B) & p(P_A = +|S_B) \\ p(S_A = +|R_B) & p(S_A = +|P_B) \end{array} \right). \quad (3)$$

Concrete values on the diagonal are irrelevant. Joint probabilities are thus given by

$$\begin{aligned}
 p(R_A = + \cap P_B = +) &= p(R_A = +|P_B)p(P_B = +|R_A) = 0 \times 1 = 0, \\
 p(R_A = - \cap P_B = +) &= p(R_A = -|P_B)p(P_B = +|R_A) = 1 \times 1 = 1, \\
 p(R_A = + \cap P_B = -) &= p(R_A = +|P_B)p(P_B = -|R_A) = 0 \times 0 = 0, \\
 p(R_A = - \cap P_B = -) &= p(R_A = -|P_B)p(P_B = -|R_A) = 1 \times 0 = 0.
 \end{aligned}$$

Similarly

$$\begin{aligned}
 p(P_A = + \cap R_B = -) &= p(S_A = + \cap P_B = -) = p(P_A = - \cap S_B = +) \\
 &= p(S_A = - \cap R_B = +) = p(R_A = + \cap S_B = -) = 1.
 \end{aligned}$$

The remaining probabilities vanish. Assuming a classical probability space, we can write

$$\begin{aligned}
 p(R_A = - \cap P_B = +) &= \mu(\mathcal{R}'_A \cap \mathcal{P}_B) = 1, \\
 p(R_A = + \cap P_B = +) &= \mu(\mathcal{R}_A \cap \mathcal{P}_B) = 0, \\
 \mu(\mathcal{R}'_A \cap \mathcal{P}_B) + \mu(\mathcal{R}_A \cap \mathcal{P}_B) &= \mu(\mathcal{P}_B) = 1.
 \end{aligned}$$

However,

$$\begin{aligned} p(S_A = + \cap P_B = +) &= \mu(\mathcal{S}_A \cap \mathcal{P}_B) = 0, \\ p(S_A = - \cap P_B = +) &= \mu(\mathcal{S}'_A \cap \mathcal{P}_B) = 0, \\ \mu(\mathcal{S}'_A \cap \mathcal{P}_B) + \mu(\mathcal{S}_A \cap \mathcal{P}_B) &= \mu(\mathcal{P}_B) = 0, \end{aligned}$$

which implies $0 = 1$. The same type of contradiction occurs in the lizard game (Section 2).

Let us now come to the ‘ace-up-one’s-sleeve’ replicator equation. The standard Kolmogorovian replicator equation,

$$\begin{aligned} \frac{dx_k}{dt} &= x_k \left(\sum_{l=1}^n a_{kl} x_l - \sum_{l,m=1}^n a_{lm} x_l x_m \right), \\ a_{lm} &= \sum_{j=1}^N b_j p_{lm;j}, \quad l, m = 1, \dots, n; \quad j = 1, \dots, N, \\ \sum_{k=1}^n x_k &= 1; \quad \sum_{j=1}^N p_{lm;j} = 1; \quad x_k \geq 0, p_{lm;j} \geq 0, \end{aligned} \tag{4}$$

involves two types of probabilities. The ones we are interested in are denoted by $p_{lm;j}$. The indices l, m in $p_{lm;j}$ index probability spaces in a probability manifold. $p_{lm;j}$ are the “pack-of-cards” probabilities implicitly present in any game, and parametrizing the game, but treated as fixed and independent of the probabilities of strategies x_k . The equivalent form of (4), found in [45], reads

$$\begin{aligned} i \frac{d\rho_1}{dt} &= [H_1(\rho_1), \rho_1], \quad \rho_1 = |1\rangle\langle 1|, \quad \langle 1| = (\sqrt{x_1}, \dots, \sqrt{x_n}), \quad x_k = \text{Tr } \rho_1 P_{1,k}, \\ H_1(\rho_1) &= i[D_1(\rho_1), \rho_1], \quad D_1(\rho_1) = \frac{1}{2} \text{diag} \left(\sum_{l=1}^n a_{1l} x_l, \dots, \sum_{l=1}^n a_{nl} x_l \right). \end{aligned}$$

Here $P_{1,k} = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$, with 1 in the k th position. Now denote $p_{11,j} = p_j$, $\langle \tilde{1}| = (\sqrt{p_1}, \dots, \sqrt{p_N})$, and $\rho_2 = |\tilde{1}\rangle\langle \tilde{1}|$. We know that there exists $E_{lm;j}$ such that $p_{lm;j} = \text{Tr } \rho_2 E_{lm;j}$. $E_{lm;j} = U_{lm} P_{2,j} U_{lm}^\dagger$ for some projectors $P_{2,j}$, $p_j = \text{Tr } \rho_2 P_{2,j}$. Define $\rho = \rho_1 \otimes \rho_2$, $\hat{a}_{lm} = \sum_j b_j \mathbb{I}_1 \otimes E_{lm;j}$. Now, $x_k = \text{Tr } \rho(P_{1,k} \otimes \mathbb{I}_2)$, $p_{lm;j} = \text{Tr } \rho(\mathbb{I}_1 \otimes E_{lm;j})$, $a_{lm} = \text{Tr } \rho \hat{a}_{lm}$. Denoting

$$\begin{aligned} D(\rho) &= \frac{1}{2} \text{diag} \left(\sum_{l=1}^n a_{1l} \text{Tr } \rho(P_{1,l} \otimes \mathbb{I}_2), \dots, \sum_{l=1}^n a_{nl} \text{Tr } \rho(P_{1,l} \otimes \mathbb{I}_2) \right) \otimes \mathbb{I}_2, \\ H(\rho) &= i[D(\rho), \rho], \end{aligned}$$

we reconstruct the standard replicator equation by taking the partial trace over the second subsystem from both sides of

$$i \frac{d\rho}{dt} = [H(\rho), \rho], \tag{5}$$

under the constraints $\rho = \rho_1 \otimes \rho_2$, $d\rho_2/dt = 0$. The constraints imply $[D(\rho), \rho] = [D_1(\rho_1), \rho_1] \otimes \rho_2$, $d\rho/dt = d\rho_1/dt \otimes \rho_2$. Relaxing the constraints in (5), one generalizes (4) to games involving players with an ace up their sleeve, where correlations between x_k and $p_{lm;j}$ are no longer ignored.

5 Conclusions

Cyclic competition, an evolutionary analogue of the RPS game, is considered to be relevant to competition dynamics in population ecology. However, modeling probabilistic RPS structures, one encounters a fundamental difficulty, i.e. the pairs of rock-scissors, scissors-paper and paper-rock possess representations in separate Kolmogorovian probability spaces, but a single global probability space for entire triplets does not exist. Populations that take part in cyclic competition should therefore involve probabilistic incompatibilities, analogous to those occurring in quantum mechanics. In this paper, we used a set of experimental data collected in the period 1990–2011 on the RPS cycles of lizards to show that these incompatibilities are indeed unavoidable and data cannot be reconstructed from a single Kolmogorovian probability space. We also prove that this effect is genuinely quantum probabilistic, i.e. all the probabilities can be formulated in terms of a single density matrix and a set of non-commuting projectors. Surprisingly, lizards occasionally engage in such competition dynamics. In consequence, the formalism of evolutionary games, similarly to quantum mechanics, should begin with density matrix equations.

Our result concerning the non-Kolmogorovian quantum-based behavior of a specific biological system with RPS dynamics resulting from cyclic competition has far-reaching consequences, since several biological systems are believed to contain sub-dynamics of the cyclic competition type [34]. For example, the coexistence of a large number of phytoplankton species competing for a limited variety of resources in aquatic ecosystems (*paradox of the plankton*) is believed to result from cyclic competition [22, 23, 24], and hence could structurally have a quantum origin.

Our investigation links up with the new developments of ‘identification of quantum structure in domains different from the micro-world’, and extends to animal behavior the successful results obtained in this field for aspects of human cognition. Furthermore, it suggests that ecological systems are intrinsically contextual systems, and constitutes a powerful support toward a systematic application of quantum-theoretical modeling in biology.

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